(M_1, g) (M_2, h) $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ $G = f_2^2 g + f_1^2 h$

$M_1 \times M_2$ f_1 f_2 M_1 M_2

1969 Bishop

[1] 2016

[2] 2018

[3] 2022

Levi-Civita Ricci [4]

[5-6]

[7] 1982 Michelsohn

[8] 1 0 [8] 2014

[9]

1985 Balas [10] Ricci

λ λ [10]

1987 Kobayashi [11]

1

(M, J, G) n J G M p

$$T_p^{\mathbb{C}} M = T_p^{1,0} M \oplus T_p^{0,1} M$$

J $\pm\sqrt{-1}.$ $z = (z^1, z^2, \cdots, z^n)$ $T_p^{1,0} M$ $(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \cdots, \frac{\partial}{\partial z^n})$

∇ G J

*

[3]

$$\Gamma^{\beta}_{\gamma\alpha} = G^{\bar{\delta}\beta}\frac{\partial G_{\alpha\bar{\delta}}}{\partial z^{\gamma}} \quad (1)$$

[12]

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$T^{\beta}_{\gamma\alpha} = \Gamma^{\beta}_{\gamma\alpha} - \Gamma^{\beta}_{\alpha\gamma} \quad (2)$$

1 3 K [11]

$$K(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

$$K^{\beta}_{\alpha\gamma\bar{\delta}} = -\frac{\partial \Gamma^{\beta}_{\gamma\alpha}}{\partial \bar{z}^{\delta}} \quad (3)$$

1 3 [11]

$$K^{\beta}_{\alpha} = G^{\bar{\delta}\gamma}K^{\beta}_{\alpha\gamma\bar{\delta}} \quad (4)$$

1[10] (M, J, G)

$$T^{\beta}_{\gamma\alpha} = 0 \quad (5)$$

(M, J, G)

2[8] (M, J, G) 1 0

$$\tau_{\gamma} = 0 \quad (6)$$

(M, J, G)

$$\tau_{\gamma} = G^{\bar{\eta}\alpha}T_{\gamma\alpha\bar{\eta}} \quad (7)$$

$$T_{\gamma\alpha\bar{\eta}} = G_{\beta\bar{\eta}}T^{\beta}_{\gamma\alpha} \quad (8)$$

3[11] (M, J, G)

$$K = \varphi I, \quad i.e., \quad K^{\beta}_{\alpha} = \varphi\delta^{\beta}_{\alpha} \quad (9)$$

φ M (M, J, G)

4[13]

$$L = G^{\bar{\beta}\alpha}\frac{\partial^2}{\partial z^{\alpha}\partial \bar{z}^{\beta}} \quad (10)$$

(M_1, g) (M_2, h) m n $M = M_1 \times M_2$ $m + n$ $\pi_1: M \to M_1$ $\pi_2: M \to M_2$ $z = (z_1, z_2) \in M, z_1 = (z^1, \cdots, z^m) \in M_1$ $z_2 = (z^{m+1}, \cdots, z^{m+n}) \in M_2$ $\pi_1(z) = z_1$ $\pi_2(z) = (z_2)$

$d\pi_1: T^{1,0}(M) \to T^{1,0}M_1, d\pi_2: T^{1,0}(M) \to T^{1,0}M_2$ π_1 π_2 $v = (v_1, v_2) \in T_z^{1,0}(M), v_1 = (v^1, \cdots, v^m) \in T_{z_1}^{1,0}M_1, v_2 = (v^{m+1}, \cdots, v^{m+n}) \in T_{z_2}^{1,0}M_2$ $d\pi_1(z,v) = (z_1, v_1)$ $d\pi_2(z,v) = (z_2, v_2)$

5[3] (M_1, g) (M_2, h) $f_1: M_1 \to (0, +\infty)$ $f_2: M_2 \to (0, +\infty)$ $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ $G: M \to (0, +\infty)$ $M = M_1 \times M_2$

$$G(z,v) = (f_2 \circ \pi_2)^2(z)g(\pi_1(z), d\pi_1(v)) + (f_1 \circ \pi_1)^2(z)h(\pi_2(z), d\pi_2(v)) \quad (11)$$

$z = (z_1, z_2) \in M, v = (v_1, v_2) \in T_z^{1,0}M$ f_1 f_2 (M_1, g) (M_2, h) $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$

$f_1 \equiv 1$ $f_2 \equiv 1$ $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ $f_1 \equiv 1$ $f_2 \equiv 1$ $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ f_1 f_2 $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$

$1 \leqslant \alpha,\beta,\gamma,\eta,\delta \leqslant m+n, 1 \leqslant i,j,k,l,t \leqslant m, m+1 \leqslant i',j',k',l',t' \leqslant m+n.$ (M_1,g) (M_2,h)

1 2 $\overset{1}{\Gamma}{}^{i}_{jk}$ $\overset{2}{\Gamma}{}^{i'}_{j'k'}$ (M_1,g) (M_2,h)

$({}_{f_2}M_1 \times {}_{f_1}M_2,G)$ (M_1,g) (M_2,h)

$$g_{i\bar{j}} = \frac{\partial^2 g}{\partial v^i \partial \overline{v^j}}, \quad h_{i'\bar{j'}} = \frac{\partial^2 h}{\partial v^{i'} \partial \overline{v^{j'}}} \tag{12}$$

G [3]

$$(G_{\alpha\bar{\beta}}) = \left(\frac{\partial^2 G}{\partial v^\alpha \partial \overline{v^\beta}}\right) = \begin{pmatrix} f_2^2 g_{i\bar{j}} & 0 \\ 0 & f_1^2 h_{i'\bar{j'}} \end{pmatrix} \tag{13}$$

$(G^{\bar{\beta}\alpha})$ [3]

$$(G^{\bar{\beta}\alpha}) = \begin{pmatrix} f_2^{-2} g^{\bar{j}i} & 0 \\ 0 & f_1^{-2} h^{\bar{j'}i'} \end{pmatrix} \tag{14}$$

2

$({}_{f_2}M_1 \times {}_{f_1}M_2,G)$

1[3] $({}_{f_2}M_1 \times {}_{f_1}M_2,G)$ $\Gamma^{\beta}_{\gamma\alpha}$

$$\Gamma^i_{jk} = \overset{1}{\Gamma}{}^i_{jk},\ \Gamma^i_{j'k} = 2f_2^{-1}\frac{\partial f_2}{\partial z^{j'}}\delta^i_k,\ \Gamma^{i'}_{jk'} = 2f_1^{-1}\frac{\partial f_1}{\partial z^j}\delta^{i'}_{k'},\ \Gamma^{i'}_{j'k'} = \overset{2}{\Gamma}{}^{i'}_{j'k'} \tag{15}$$

$$\Gamma^i_{jk'} = \Gamma^i_{j'k'} = \Gamma^{i'}_{jk} = \Gamma^{i'}_{j'k} = 0 \tag{16}$$

1 $({}_{f_2}M_1 \times {}_{f_1}M_2,G)$ $T^{\beta}_{\gamma\alpha}$

$$T^i_{jk} = \overset{1}{T}{}^i_{jk},\ T^{i'}_{j'k'} = \overset{2}{T}{}^{i'}_{j'k'} \tag{17}$$

$$T^i_{j'k} = 2f_2^{-1}\frac{\partial f_2}{\partial z^{j'}}\delta^i_k, \quad T^i_{jk'} = -2f_2^{-1}\frac{\partial f_2}{\partial z^{k'}}\delta^i_j \tag{18}$$

$$T^{i'}_{j'k} = -2f_1^{-1}\frac{\partial f_1}{\partial z^k}\delta^{i'}_{j'}, \quad T^{i'}_{jk'} = 2f_1^{-1}\frac{\partial f_1}{\partial z^j}\delta^{i'}_{k'} \tag{19}$$

$$T^i_{j'k'} = T^{i'}_{jk} = 0 \tag{20}$$

2 $\alpha = k, \beta = i, \gamma = j$ 15

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj} = \overset{1}{\Gamma}{}^i_{jk} - \overset{1}{\Gamma}{}^i_{kj} = \overset{1}{T}{}^i_{jk}$$

1 $({}_{f_2}M_1 \times {}_{f_1}M_2,G)$ $({}_{f_2}M_1 \times {}_{f_1}M_2,G)$ (M_1,g) (M_2,h) f_1 f_2

1 $({}_{f_2}M_1 \times {}_{f_1}M_2,G)$ $T^{\beta}_{\gamma\alpha} = 0$ 1

$$\begin{cases} \overset{1}{T}{}^i_{jk} = 0 \\ \overset{2}{T}{}^{i'}_{j'k'} = 0 \\ \dfrac{\partial f_1}{\partial z^k} = 0 \\ \dfrac{\partial f_2}{\partial z^{k'}} = 0 \end{cases} \tag{21}$$

21 (M_1, g) (M_2, h) 21

f_1 f_2

2 $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ $T_{\gamma\alpha\bar{\eta}}$

$$T_{jk\bar{l}} = f_2^2 \overset{1}{T}_{jk\bar{l}},\ T_{j'k'\bar{l}'} = f_1^2 \overset{2}{T}_{j'k'\bar{l}'} \tag{22}$$

$$T_{jk'\bar{l}} = -2f_2\frac{\partial f_2}{\partial z^{k'}} g_{j\bar{l}},\ T_{jk'\bar{l}'} = 2f_1\frac{\partial f_1}{\partial z^j} h_{k'\bar{l}'} \tag{23}$$

$$T_{j'k\bar{l}} = 2f_2\frac{\partial f_2}{\partial z^{j'}} g_{k\bar{l}},\ T_{j'k\bar{l}'} = -2f_1\frac{\partial f_1}{\partial z^k} h_{j'\bar{l}'} \tag{24}$$

$$T_{jk\bar{l}'} = T_{j'k'\bar{l}} = 0 \tag{25}$$

8 $\alpha = k, \gamma = j, \eta = l$

$$T_{jk\bar{l}} = G_{\beta\bar{l}}T^{\beta}_{jk} = G_{i\bar{l}}T^{i}_{jk} + G_{i'\bar{l}}T^{i'}_{jk} \tag{26}$$

17 13 26

$$T_{jk\bar{l}} = f_2^2 g_{i\bar{l}}\overset{1}{T}{}^{i}_{jk} = f_2^2\overset{1}{T}_{jk\bar{l}}$$

3 $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ τ_γ

$$\tau_j = \overset{1}{\tau}_j + 2f_1^{-1}\frac{\partial f_1}{\partial z^j} \tag{27}$$

$$\tau_{j'} = \overset{2}{\tau}_{j'} + 2f_2^{-1}\frac{\partial f_2}{\partial z^{j'}} \tag{28}$$

7 $\gamma = j$

$$\tau_j = G^{\bar{\eta}\alpha}T_{j\alpha\bar{\eta}} = G^{\bar{l}k}T_{jk\bar{l}} + G^{\bar{l}k'}T_{jk'\bar{l}} + G^{\bar{l}'k}T_{jk\bar{l}'} + G^{\bar{l}'k'}T_{jk'\bar{l}'} \tag{29}$$

14 2 29

$$\begin{aligned}\tau_j &= f_2^{-2}g^{\bar{l}k}f_2^2\overset{1}{T}_{jk\bar{l}} + f_1^{-2}h^{\bar{l}'k'}2f_1\frac{\partial f_1}{\partial z^j}h_{k'\bar{l}'}\\ &= g^{\bar{l}k}\overset{1}{T}_{jk\bar{l}} + 2f_1^{-1}\frac{\partial f_1}{\partial z^j}\\ &= \overset{1}{\tau}_j + 2f_1^{-1}\frac{\partial f_1}{\partial z^j}\end{aligned}$$

28

2 $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$

$$\begin{cases}\overset{1}{\tau}_j + 2f_1^{-1}\dfrac{\partial f_1}{\partial z^j} = 0\\ \overset{2}{\tau}_{j'} + 2f_2^{-1}\dfrac{\partial f_2}{\partial z^{j'}} = 0\end{cases} \tag{30}$$

2 $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ $\tau_\gamma = 0$

$$\begin{cases}\tau_j = 0\\ \tau_{j'} = 0\end{cases} \tag{31}$$

27 28 31 30

1 $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ f_1 f_2 $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$

(M_1, g) (M_2, h)

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2[3] (

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